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Manifest supersymmetry in non-commutative geometry

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Abstract: We consider the open superstring ending on a D-brane in the presence of a constant NS-NS B field, using the Green-Schwarz formalism. Quantizing in the light-cone gauge, we find that the anti-commutation relations for the fermionic variables of superspace remain unmodified. We also derive the unbroken supersymmetry algebra living on the D-brane. This establishes how the Moyal product is extended in a superspace formulation of non-commutative field theories. The superfield formulation of non-commutative supersymmetric field theories is briefly considered.

Keywords: D-branes, Space-Time Symmetries, Supersymmetry and Duality, Superspaces.

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Contents

1. Introduction

During the last years, there has been remarkable progress in understanding the dynamics of supersymmetric gauge theories and string theory. In fact, both fields have benefited from each other, in elaborate and beautiful ways. Recently, a new connection between them has arisen: string theory backgrounds with a constant NS-NS B field correspond, in the Sen-Seiberg limit [3], to non-commutative gauge theories [1, 2]. The dynamics of these theories appears up to now quite mysterious, in particular at distances comparable to the non-commutativity scale. Looking back to the examples of ordinary supersymmetric gauge theories, it seems quite possible that supersymmetry will provide strong constraints on the dynamics of these non-commutative field theories.

Therefore, it seems worthwhile to analyze how supersymmetry is realized in these theories. To this end, we would like to work in a context where supersymmetry is as manifest as possible. For certain numbers of spacetime dimensions and supersymmetries, one can construct a superspace and superfield formalism for ordinary supersymmetric field theories. It is very likely that a similar construction is available for non-commutative supersymmetric field theories, where some additional fermionic variables promote a given manifold to a supermanifold. But then, as a mathematical question in the context of non-commutative geometry, one could imagine different consistent deformations of the fermionic anti-commutation rules. For example, a possible deformation of the multiplication between fermionic coordinates of superspace could be

$$f(\theta) \star g(\theta) = e^{\frac{\Delta^{\alpha\beta}}{2}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta'^\beta}} f(\theta)g(\theta')\Big|_{\theta=\theta'}, \tag{1.1}$$

where $\Delta^{\alpha\beta}$ is symmetric. An explicit deformation of $N = 1$ superspace in two dimensions has been considered in [4]. Quantum group [5] like deformations may also be envisaged. To find out what happens in string theory is one of the purposes of this letter.

In previous papers [6, 7, 8, 9], the quantization of an open string ending on a D-brane with a constant NS-NS B-field background was carried out in the RNS formulation. The result was that the spacetime coordinates on the world-volume of the D-brane were non-commutative. In this paper we analyze the same situation, but with the string formulated in the Green-Schwarz [10] language.[1] We will take advantage of the manifest spacetime supersymmetry in this formalism to obtain the anti-commutation relations for the grassmanian variables θ of superspace, in a situation where the bosonic coordinates X do not commute. The result is that the fermionic coordinates are *undeformed*, and have the same anti-commutation rules on the brane. Also, we will derive the supersymmetry algebra living on the D-brane and verify that its structure is not modified. Hence, if we want to work with superfields evaluated on flat superspace, their non-commutative associative $*$-product reduces to the ordinary anti-commuting product for the fermionic variables θ of superspace.

The paper is organized as follows. We start in section 2, reviewing the construction of the Green-Schwarz (GS) superstring action in a general type-II supergravity background; after that, we deal with the case of interest: a flat background with a constant NS-NS B field, where the D-brane boundary conditions are established and solved. In section 3 we carry out the light-cone quantization of the GS superstring; we compute the unbroken supersymmetry algebra living on the world-volume of the D-brane, finding the same structure as in the ordinary (commutative) theories; we also show how the appearance of the Moyal product can be understood from the oscillator quantization method. Finally, in section 4 we introduce the notions of superspace and superfields in non-commutative geometry. The WZ model is discussed as an illustrative example, and we observe that the perturbative non-renormalization theorem for the F-term is still valid.

2. D-branes in the covariant Green-Schwarz formalism

2.1 Green-Schwarz superstring in a general type-II background

The first thing we need to know is how the GS superstring couples to the B-field. In this subsection we review the construction of [13], which ensures a κ-invariant action.

[1] The analysis of D-branes from the Green-Schwarz superstring, without a background B-field, has been considered in [11, 12]

Let us start by setting up some conventions. The ten dimensional $\mathcal{N} = 2$ superspace is a supermanifold of dimension $(10|32)$. Since we will first present the GS superstring propagating in a general type-II supergravity background, we denote by $Z^{\underline{M}} = (X^{\underline{\mu}}, \theta^m)$ its local coordinates and by $\underline{A} = (\underline{a}, \alpha)$ its target space tangent indices. Underlining an index indicates that it is 10 dimensional.[2] The super-vielbeins are denoted by $E_{\underline{M}}{}^{\underline{A}}$. The fermionic variables θ are spacetime spinors and consist of two Majorana-Weyl spinors. In type-IIA superspace, these spinors carry opposite chiralities, which can be combined into a single 32 component Majorana spinor; while in type IIB they are of the same chirality. We will use the fermionic index α for both cases. In type IIA, it is a 32 dimensional spinor index, while in type IIB it is understood to be a composite index of a Majorana spinor index (32 dimensional) and an SO(2) doublet index acted on by the Pauli matrices. Chiral projection operators should be inserted in appropriate places to reduce the Majorana indices to Majorana-Weyl ones.

The covariant κ-symmetric action for the $N = 2$ superstring in a supergravity background takes the form [13]

$$I = -\frac{1}{2\pi\alpha'} \int d^2\xi \, \left\{ \sqrt{-g} g^{ij} E_i{}^{\underline{a}} E_j{}^{\underline{b}} \eta_{\underline{ab}} - \epsilon^{ij} E_i{}^{\underline{B}} E_j{}^{\underline{A}} B_{\underline{AB}} \right\}, \tag{2.1}$$

where g_{ij} is the metric of the world-sheet, $\epsilon^{01} = 1$, $\eta_{ab} = \mathrm{diag}(-,+,\ldots,+)$ is the 10d Minkowski spacetime metric and the pull-back super-vielbeins are

$$E_i{}^{\underline{A}} \equiv \partial_i Z^{\underline{M}} E_{\underline{M}}{}^{\underline{A}} = (E_i{}^{\underline{a}}, E_i{}^{\alpha}) \, . \tag{2.2}$$

What is relevant for us is the second term in (2.1), the pull-back of the 2-form superfield

$$B = \frac{1}{2} E^{\underline{B}} E^{\underline{A}} B_{\underline{AB}} \, , \tag{2.3}$$

which is the (local) potential of the closed super 3-form $H = dB$ of type-II supergravity. Since we will consider world-sheets with boundaries, properly there are also the vector superfields A with supersymmetric couplings to the string [14]. But in order to simplify the discussion, and without any loss of generality for the case of constant B-field, we confine to the gauge where A is zero.

The action (2.1) is invariant under the κ-symmetry:

$$\begin{aligned} \delta_\kappa Z^{\underline{a}} &= 0 \, , \\ \delta_\kappa Z^{\alpha} &= \frac{1}{2} \kappa^{\gamma}(\xi)(1 + \Gamma)_{\gamma}{}^{\alpha} \, , \end{aligned} \tag{2.4}$$

[2] We are following basically the notation and conventions of [14] for the super-embedding, except that, in order to simplify the notation, we will not distinguish whether a fermionic index is 10 dimensional or $(p+1)$-dimensional (for a Dp-brane) and so we will not underline the fermionic indices. This is sufficient for our purposes, as we will not need to split the 10-dimensional Γ-matrices into those appropriate for the $(p+1)$-dimensional submanifold. All the Γ-matrices in this paper are the ten dimensional ones.

JHEP02(2000)022

with the definitions $\delta_\kappa Z^{\underline{A}} := \delta_\kappa Z^{\underline{M}} E_{\underline{M}}{}^{\underline{A}}$ and

$$(\Gamma)_\alpha{}^\beta = \frac{1}{2\sqrt{-g}} \epsilon^{ij} \left(E_i{}^{\underline{a}} E_j{}^{\underline{b}} \Gamma_{\underline{ab}} P\right)_\alpha{}^\beta ,$$
$$P = \begin{cases} \Gamma_{11} & \text{(IIA)} \\ \sigma_1 & \text{(IIB)} \end{cases} . \tag{2.5}$$

Note that Γ_{11} acts on the 32-component Majorana index, while σ_3 acts on the SO(2) doublet index of the two 16-component Majorana-Weyl spinors of type IIB.

The constraints which follow by demanding κ-symmetry invariance of (2.1) correspond to the ones of the 10d $\mathcal{N} = 2$ supergravity multiplet [15]. There are the constraints on the torsion super two-form,

$$T_{\alpha\beta}{}^{\underline{a}} = -2i(\Gamma^{\underline{a}})_{\alpha\beta} ,$$
$$T_{\alpha\underline{b}}{}^{\underline{a}} = \delta_{\underline{b}}{}^{\underline{a}} \, \chi_\alpha , \tag{2.6}$$

and on the super three-form,

$$H_{\alpha\beta\gamma} = 0 ,$$
$$H_{\alpha\beta\underline{a}} = 2i(\Gamma_{\underline{a}} Q)_{\alpha\beta} ,$$
$$H_{\alpha\underline{ab}} = (\Gamma_{\underline{ab}} Q\chi)_\alpha , \tag{2.7}$$

where

$$Q = \begin{cases} \Gamma_{11} & \text{(IIA)} \\ -\sigma_3 & \text{(IIB)} \end{cases} \tag{2.8}$$

and χ_α is a spinor superfield [16] proportional to the dilaton superfield of the supergravity background, i.e. $\chi \propto D\phi$.

2.2 Constant B-field in flat spacetime

We are interested in type-II backgrounds with flat 10 dimensional Minkowski spacetime and a constant B-field. In this case, the dilaton is constant and $\chi = 0$. With no distinction between curved and tangent indices, the torsion constraints (2.6) are satisfied by the flat space pull-back,

$$E_i{}^{\underline{\mu}} = \partial_i X^{\underline{\mu}} - i\bar{\theta}\Gamma^{\underline{\mu}}\partial_i\theta ,$$
$$E_i{}^\alpha = \partial_i \theta^\alpha . \tag{2.9}$$

Equally, the 3-form constraints (2.7) are solved by the super 2-form B_{AB} with the non-zero components

$$B_{\underline{\mu\nu}} = \text{constant} ,$$
$$B_{\underline{\mu}\alpha} = i(\Gamma_{\underline{\mu}} Q\theta)_\alpha + iB_{\underline{\mu\nu}}(\Gamma^{\underline{\nu}}\theta)_\alpha ,$$
$$B_{\alpha\beta} = -(\Gamma_{\underline{\mu}}\theta)_{(\alpha}(\Gamma^{\underline{\mu}} Q\theta)_{\beta)} - B_{\underline{\mu\nu}}(\Gamma^{\underline{\mu}}\theta)_\alpha(\Gamma^{\underline{\nu}}\theta)_\beta . \tag{2.10}$$

Substituting the expressions (2.9) and (2.10) into the action (2.1), we find

$$I = -\frac{1}{2\pi\alpha'}\int d^2\xi\Big\{\sqrt{-g}g^{ij}\Pi_i{}^{\underline{\mu}}\Pi_j{}^{\underline{\nu}}\eta_{\underline{\mu}\underline{\nu}} + 2i\epsilon^{ij}\partial_i X^{\underline{\mu}}\left(\bar\theta^1\Gamma_{\underline{\mu}}\partial_j\theta^1 - \bar\theta^2\Gamma_{\underline{\mu}}\partial_j\theta^2\right) - \\ - 2\epsilon^{ij}(\bar\theta^1\Gamma^{\underline{\mu}}\partial_i\theta^1)(\bar\theta^2\Gamma_{\underline{\mu}}\partial_j\theta^2) + \epsilon^{ij}\partial_i X^{\underline{\mu}}\partial_j X^{\underline{\nu}}B_{\underline{\mu}\underline{\nu}}\Big\}\,, \tag{2.11}$$

where we have introduced the more common notation $(E_i{}^{\underline{\mu}} \to)\ \Pi_i{}^{\underline{\mu}} = \partial_i X^{\underline{\mu}} - i\bar\theta^A\Gamma\theta^A$ with θ^A $(A=1,2)$ the two Majorana-Weyl spinors of type II. We have also used in the above that $\Gamma_{11} = -1$ for θ^1 ($+1$ for θ^2) in IIA and $\sigma_3 = 1$ for θ^1 (-1 for θ^2) in IIB. Since (2.9) and (2.10) satisfy the constraints (2.6) and (2.7), the action (2.11) is guaranteed to be κ-symmetric.

Thus, we see that for a constant B-field, the only effect on the action is the addition of the last term in (2.11), the usual bosonic coupling to the target space. A coupling to the fermionic spacetime variables θ^A would arise only if B were not closed. The couplings to boundary fields can be discussed using the D-brane vertex operators.

2.3 D-brane boundary conditions

We split the 10d indices $\underline{\mu} = 0,1,\ldots,9$, into $\mu = 0,1,\ldots,p$ for the directions longitudinal to the Dp-brane world-volume[3] and $\mu' = p+1,\ldots,9$ for its transverse directions. The components $B_{\mu\nu}$ along the direction of the brane are non-zero in the chosen gauge $A=0$. The remaining components, $B_{\underline{\mu}\nu'}$, are set to zero by a gauge transformation. Since $B_{\mu\nu}$ is constant, the equations of motion take the same form as in the case with zero B-field,

$$\begin{aligned}\Pi_i\cdot\Pi_j - \frac{1}{2}g_{ij}g^{kl}\Pi_k\cdot\Pi_l &= 0\,,\\ \Gamma\cdot\Pi_i P_{-}{}^{i}{}_{j} g^{jk}\partial_k\theta^1 &= 0\,,\\ \Gamma\cdot\Pi_i P_{+}{}^{i}{}_{j} g^{jk}\partial_k\theta^2 &= 0\,,\\ \partial_i\Big\{\sqrt{-g}g^{ij}\partial_j X^\mu - 2iP_{-}{}^{i}{}_{j}g^{jk}\bar\theta^1\Gamma^\mu\partial_k\theta^1 - 2iP_{+}{}^{i}{}_{j}g^{jk}\bar\theta^2\Gamma^\mu\partial_k\theta^2\Big\} &= 0\,,\end{aligned} \tag{2.12}$$

where $P_{\pm}{}^{i}{}_{j} = \frac{1}{2}(\delta^i_j \pm \epsilon^{ik}g_{kj}/\sqrt{-g})$.

For the boundary terms, however, the B field matters. In the world-sheet conformal gauge, with coordinates $\xi^i = \{\tau,\sigma\}$, the variation of the action (2.11) requires the following boundary contributions to be zero:

$$\delta X_{\underline{\mu}}\left(\Pi_\sigma{}^{\underline{\mu}} - i\bar\theta^1\Gamma_{\underline{\mu}}\partial_\tau\theta^1 + i\bar\theta^2\Gamma_{\underline{\mu}}\partial_\tau\theta^2\right) + \delta X_\mu\partial_\tau X^\nu B_\nu{}^\mu + i(\bar\theta^A\Gamma_{\underline{\mu}}\delta\theta^A)\Pi_\sigma{}^{\underline{\mu}} + \tag{2.13}$$
$$+ i\partial_\tau X^{\underline{\mu}}(\bar\theta^1\Gamma_{\underline{\mu}}\delta\theta^1 - \bar\theta^2\Gamma_{\underline{\mu}}\delta\theta^2) + (\bar\theta^1\Gamma^{\underline{\mu}}\delta\theta^1\bar\theta^2\Gamma_{\underline{\mu}}\partial_\tau\theta^2 - \bar\theta^2\Gamma^{\underline{\mu}}\delta\theta^2\bar\theta^1\Gamma_{\underline{\mu}}\partial_\tau\theta^1)|_{\sigma=0,\pi} = 0\,.$$

[3]For convenience in the light-cone gauge, in section 3 we will take the 9-direction to be always longitudinal to the D-brane.

The δX terms and the $\delta\theta$ terms must vanish independently. Since $\delta X^\mu = \delta\tau\partial_\tau X^\mu \neq 0$, the vanishing of all the $\delta X_{\underline{\mu}}$ terms requires the following boundary conditions ($\sigma = 0, \pi$):

$$\partial_\tau X^{\mu'} = 0\,, \qquad \mu' = p+1, \ldots, 9\,, \tag{2.14}$$

$$\partial_\sigma X^\mu + \partial_\tau X^\nu B_\nu{}^\mu = 0, \qquad \mu = 0, \ldots, p\,, \tag{2.15}$$

$$\bar\theta^1\Gamma^\mu(\partial_\sigma + \partial_\tau)\theta^1 + \bar\theta^2\Gamma^\mu(\partial_\sigma - \partial_\tau)\theta^2 = 0\,. \tag{2.16}$$

We will comment on the last condition later. Using (2.14) and (2.15), the $\delta\theta$ terms reduce to ($\sigma = 0, \pi$)

$$\begin{aligned} i\partial_\sigma X^{\mu'}\left(\bar\theta^1\Gamma_{\mu'}\delta\theta^1 + \bar\theta^2\Gamma_{\mu'}\delta\theta^2\right) + i\partial_\tau X^\mu\left[(1-B)_{\mu\nu}\bar\theta^2\Gamma^\nu\delta\theta^2 - (1+B)_{\mu\nu}\bar\theta^1\Gamma^\nu\delta\theta^1\right] + \\ + \bar\theta^2\Gamma_{\underline{\mu}}\delta\theta^2\bar\theta^2\Gamma^{\underline{\mu}}\partial_\tau\theta^2 - \bar\theta^1\Gamma_{\underline{\mu}}\delta\theta^1\bar\theta^1\Gamma^{\underline{\mu}}\partial_\tau\theta^1\,. \end{aligned} \tag{2.17}$$

This can be made to vanish if we relate the two Majorana-Weyl spinors at the ends of the open string, by imposing

$$\theta^2 = \Gamma_B\theta^1 \qquad \text{at } \sigma = 0, \pi\,, \tag{2.18}$$

where Γ_B is the matrix

$$\Gamma_B = \begin{cases} e^{(\frac{1}{2}Y_{\mu\nu}\Gamma^{\mu\nu}\Gamma_{11})}(\Gamma_{11})^{\frac{p-2}{2}}\Gamma^{0\cdots p} & \text{IIA} \\ e^{(-\frac{1}{2}Y_{\mu\nu}\Gamma^{\mu\nu}\sigma_3)}(\sigma_3)^{\frac{p-3}{2}}i\sigma_2\,\Gamma^{0\cdots p} & \text{IIB} \end{cases} \tag{2.19}$$

with

$$Y = \frac{1}{2}\ln\left(\frac{1+B}{1-B}\right). \tag{2.20}$$

Indeed, let C be the charge conjugation matrix C defined by $\Gamma^{\mu T} = -C\Gamma^\mu C^{-1}$. The conjugate spinor $\bar\theta = \theta^T C$ then satisfies $\bar\theta^2 = \bar\theta^1\hat\Gamma$ at $\sigma = 0, \pi$, where $\hat\Gamma_B = C^{-1}\Gamma_B^T C$. It is easy to verify that

$$\hat\Gamma_B\Gamma^{\mu'} = -\Gamma^{\mu'}\Gamma_B^{-1} \tag{2.21}$$

and

$$\hat\Gamma_B\Gamma^\mu = \begin{cases} (e^{2Y\Gamma_{11}})^\mu{}_\nu\Gamma^\nu\Gamma_B^{-1}\,, & \text{IIA} \\ (e^{2Y\sigma_3})^\mu{}_\nu\Gamma^\nu\Gamma_B^{-1}\,, & \text{IIB} \end{cases} \tag{2.22}$$

In particular, since $\Gamma_{11} = +1$ for θ^2 (-1 for θ^1) in IIA and $-\sigma_3 = +1$ for θ^2 (-1 for θ^1) in IIB, we have

$$\hat\Gamma_B\Gamma^\mu\theta^2 = \left(\frac{1+B}{1-B}\right)^\mu{}_\nu\Gamma^\nu\theta^1, \quad \sigma = 0, \pi\,, \tag{2.23}$$

in both IIA and IIB. The relations (2.18), (2.21) and (2.23) guarantee that the first line of (2.17) vanishes. The second line also vanishes by using (2.23) twice.

Observe that with the boundary condition (2.18), the action (2.11) is now invariant under the spacetime supersymmetry transformations,

$$
\begin{aligned}
\delta X^{\underline{\mu}} &= i\bar{\epsilon}^A \Gamma^{\underline{\mu}} \theta^A \,, \\
\delta\theta^A &= \epsilon^A
\end{aligned}
\tag{2.24}
$$

only when the constant spinors ϵ^A are related by

$$
\epsilon^2 = \Gamma_B \epsilon^1 \,. \tag{2.25}
$$

One can easily verify that $\Gamma_B^2 = 1$, and therefore the previous equation completely determines one Majorana-Weyl spinor in terms of the other. It only leaves 16 independent supersymmetry transformations linearly realized on the D-brane. These are exactly the ones found in [17], where the matrix Γ_B appeared in the κ-symmetry transformations for the supersymmetric Born-Infeld action of a Dp-brane, in a background where the gauge invariant combination $\mathcal{F} = dA - B$ was non vanishing.[4] Here we derived the same condition from the open string. We also note that, for the particular case of the D3-brane, $\epsilon^2 = -i\ \mathrm{sign}(\mathrm{Pf}(B))\epsilon^1$ when $B \to \infty$. The additional factor of i we have, as compared to [2], is due to the minkowskian metric.

One can also check that the boundary conditions are compatible with the supersymmetry (2.24), (2.25). Supersymmetric variation of (2.14) gives

$$
\delta X^{\mu'} = i\bar{\epsilon}^A \Gamma^{\mu'} \theta^A = 0 \tag{2.26}
$$

using (2.21); while that of (2.15) gives

$$
\delta(\partial_\sigma X^\mu + \partial_\tau X^\nu B_\nu{}^\mu) = -2i\bar{\epsilon}^1 \left(\frac{B}{1-B}\right)^\mu{}_\nu \Gamma^\nu (\partial_\sigma + \partial_\tau)\theta^1 \,, \tag{2.27}
$$

where we have used $\partial_\sigma\theta^2 = -\Gamma_B \partial_\sigma \theta^1$ which follows from (2.18) and the equations of motion (2.12). Hence, the boundary conditions are supersymmetric if we also impose

$$
(\partial_\sigma + \partial_\tau)\theta^1 = 0 \,, \qquad \sigma = 0, \pi \,. \tag{2.28}
$$

This, together with (2.16), gives

$$
(\partial_\sigma - \partial_\tau)\theta^2 = 0 \,, \qquad \sigma = 0, \pi \,. \tag{2.29}
$$

Equations (2.28) and (2.29) are nothing but part of the equations of motion in light-cone gauge and thus imposing these boundary conditions is consistent with light-cone gauge fixing.

[4]With the slight difference that our $B_{\mu\nu}$ is their $-i\mathcal{F}_{\mu\nu}$ due to our normalization of the B-field.

3. Light-cone gauge

The covariant quantization of the Green-Schwarz superstring encounters serious obstacles. On the other hand, the quantization in light-cone gauge is as straightforward as in the RNS formalism, with the advantage that spacetime supersymmetry is manifest.

Taking the directions 0 and 9 to be along the D-brane,[5] the light-cone gauge is defined by

$$X^+(\sigma,\tau) = x^+ + \alpha' p^+ \tau \,, \tag{3.1}$$

$$\Gamma^+ \theta^A = 0 \,, \qquad A = 1, 2 \,, \tag{3.2}$$

where $X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^9)$ and $\Gamma^\pm \equiv \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9)$. Observe that (3.2) is compatible with the boundary condition (2.18), assuming that $B_{0\mu} = B_{9\mu} = 0$.

3.1 Quantization

In light-cone gauge, only an SO(8) symmetry is apparent. The surviving components of θ

$$S^A = \sqrt{p^+}\Gamma^- \theta^A \tag{3.3}$$

are in the two inequivalent (same) eight-dimensional spinorial representations of spin(8) for type IIA (IIB), respectively. Our ten-dimensional Γ-matrices satisfy $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$, with $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. We consider the particular realization ($I = 1, \ldots, 8$)

$$\Gamma^0 = \begin{pmatrix} 0 & -1_{16} \\ 1_{16} & 0 \end{pmatrix}, \qquad \Gamma^I = \begin{pmatrix} 0 & \hat{\gamma}^I \\ \hat{\gamma}^I & 0 \end{pmatrix}, \qquad \Gamma^9 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix},$$
$$\Gamma^{11} = \begin{pmatrix} 1_{16} & 0 \\ 0 & -1_{16} \end{pmatrix}, \qquad \hat{\gamma}^I = \begin{pmatrix} 0 & \gamma^I_{a\dot{a}} \\ \gamma^I_{\dot{a}a} & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}, \tag{3.4}$$

where $\gamma^I_{a\dot{a}}$ are the Clebsch-Gordan coefficients for the coupling of the three inequivalent eight-dimensional representations of spin(8) [10]. In this basis, the Majorana-Weyl spinor θ^1 is given by[6]

$$\theta^1 = -\frac{1}{\sqrt{2p^+}} \begin{pmatrix} 0 \\ 0 \\ S^{(1)a} \\ 0 \end{pmatrix}. \tag{3.5}$$

[5]From now on, our discussion will only apply for $p > 0$.

[6]To avoid possible confusion, we will adopt the notation of putting a parenthesis around the SO(2) indices A, when we want to refer to the eight-dimensional real spinors, $S^{(A)}$, of spin(8).

Then, the fermion boundary condition (2.18) becomes

$$S^{(2)} = N \cdot S^{(1)}\,, \qquad \text{at } \sigma = 0, \pi\,, \tag{3.6}$$

where

$$N_{\dot a b} = (e^{\frac{1}{2}Y_{ij}\gamma^{ij}}\gamma^{p\cdots 8})_{\dot a b}\,, \qquad \text{for IIA} \tag{3.7}$$

and

$$N_{ab} = (e^{\frac{1}{2}Y_{ij}\gamma^{ij}}\gamma^{p\cdots 8})_{ab}\,, \quad \text{for IIB}\,. \tag{3.8}$$

In deriving these relations, we have taken advantage of $\Gamma_{11}\theta^1 = -\theta^1$, in both type IIA and IIB, to insert Γ_{11} on the right hand side of Γ_B in (2.18) in order to convert $\Gamma^{01\cdots(p-1)9}$ to $\Gamma^{p\cdots 8}$.

The SO(8) triality allows us to clarify the meaning of N. The boundary conditions for the bosonic fields can be written as usual,

$$\partial X^I = M^{IJ}\bar\partial X^J \quad \text{at } \sigma = 0, \pi\,, \tag{3.9}$$

with $M^{IJ} = (M^{ij}, M^{i'j'})$ given by

$$M^{ij} = \left(\frac{1-B}{1+B}\right)^{ij}, \quad i = 1, \ldots, p-1 \qquad \text{and} \quad M^{i'j'} = -\delta^{i'j'}, \quad i' = p, \ldots, 8\,. \tag{3.10}$$

M^{IJ} is an O(8) orthogonal matrix acting on the $\mathbf{8_v}$ representation. The eight-dimensional orthogonal matrix N_{ab} is the corresponding matrix acting on the $\mathbf{8_s}$ representation of $S^{(1)}$. Let $N_{\dot a\dot b} = (e^{\frac{1}{2}Y_{ij}\gamma^{ij}}\gamma^{p\cdots 8})_{\dot a\dot b}$, be the one for the $\mathbf{8_c}$ representation. They follow

$$\gamma^I_{a\dot a}M^{IJ} = N_{ab}N_{\dot a\dot b}\gamma^J_{b\dot b}\,. \tag{3.11}$$

A similar relation can be written for IIA. Also, it is easy to verify that N is orthogonal,

$$N_{ab}N_{ac} = \delta_{bc}\,, \qquad N_{\dot a b}N_{\dot a c} = \delta_{bc}\,. \tag{3.12}$$

In light-cone gauge, the superstring action (2.11) takes the very simple form

$$I_{\text{l.c.}} = \frac{1}{2\pi\alpha'}\int d\sigma d\tau \Big\{ -(\partial_\tau X)^2 + (\partial_\sigma X)^2 + 2\partial_\tau X^i \partial_\sigma X^j B_{ij} + \\ + i\alpha' S^{(1)}\bar\partial S^{(1)} + i\alpha' S^{(2)}\partial S^{(2)} \Big\}\,, \tag{3.13}$$

where we have introduced the left-moving and right-moving derivatives $\partial = \frac{1}{2}(\partial_\tau - \partial_\sigma)$ and $\bar\partial = \frac{1}{2}(\partial_\tau + \partial_\sigma)$. The equations of motion (2.12) become the free equations

$$\partial\bar\partial X^I = 0\,, \qquad \bar\partial S^{(1)} = 0\,, \qquad \partial S^{(2)} = 0\,. \tag{3.14}$$

With the boundary conditions (3.6) and (3.9), the mode expansions of the fields are[7]

$$X^i(\tau,\sigma) = x_0^i + \alpha'(p_0^i\tau - p_0^j B_j{}^i\sigma) + \sqrt{\alpha'}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(ia_n^i\cos(n\sigma) - a_n^j B_j{}^i\sin(n\sigma)\right), \qquad i = 1,\ldots,p-1\,, \tag{3.15}$$

$$X^{i'}(\tau,\sigma) = x_0^{i'} + \sqrt{\alpha'}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}a_n^{i'}\sin(n\sigma)\,, \qquad i' = p,\ldots,8\,, \tag{3.16}$$

$$\begin{aligned} S^{(1)}(\tau-\sigma) &= \sum_n S_n e^{-in(\tau-\sigma)}\,,\\ S^{(2)}(\tau+\sigma) &= \sum_n NS_n e^{-in(\tau+\sigma)}\,. \end{aligned} \tag{3.17}$$

The hamiltonian of the string in light-cone gauge is

$$\begin{aligned} H &= \frac{1}{2\pi p^+\alpha'^2}\int_0^\pi d\sigma\left\{(\partial_\tau X)^2 + (\partial_\sigma X)^2 - i\alpha' S^{(1)}\partial_\sigma S^{(1)} + i\alpha' S^{(2)}\partial_\sigma S^{(2)}\right\}\\ &= \frac{1}{2p^+}\left(p^ip^i + \frac{1}{\alpha'}\sum_{n\neq 0}\left(\alpha_n^i\alpha_{-n}^j + \alpha_n^{i'}\alpha_{-n}^{i'} + 2nS_n^aS_{-n}^a\right)\right), \end{aligned} \tag{3.18}$$

where $B_{ij}^2 = B_i{}^k B_{kj}$, and we have introduced $p^i = p_0^j(1-B)_j{}^i$, $\alpha_n^i = a_n^j(1-B)_j{}^i$, $\alpha_n^{i'} = a_n^{i'}$. The transverse momentum is zero and does not appear in the hamiltonian.

Following the methods of [6], for those modes living on the world-volume of the D-brane, one obtains the (anti) commutation relations

$$[x_0^i, p^j] = 2i\left(\frac{1}{1+B}\right)^{ij}, \qquad [p^i,p^j] = 0\,, \tag{3.19}$$

$$[x_0^i, x_0^j] = 2\pi i\alpha'\left(\frac{B}{1-B^2}\right)^{ij}, \tag{3.20}$$

$$[\alpha_n^i, \alpha_m^j] = 2n\delta^{ij}\delta_{n+m}\,, \tag{3.21}$$

$$\{S_n^a, S_m^b\} = \delta^{ab}\delta_{n+m}\,, \tag{3.22}$$

$$[x_0^i, \alpha_n^j] = [x_0^i, S_n^a] = [\alpha_n^i, S_m^a] = 0\,. \tag{3.23}$$

The modification in the bosonic modes, x_0^i and α_n^i, introduces the notion of non-commutative space for the world-volume of the D-brane, since one can verify that[8]

$$[X^i(\tau,\sigma), X^j(\tau,\sigma')] = \begin{cases} 2\pi i\alpha'\left(\dfrac{B}{1-B^2}\right)^{ij}, & \sigma=\sigma'=0\\ -2\pi i\alpha'\left(\dfrac{B}{1-B^2}\right)^{ij}, & \sigma=\sigma'=\pi\\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.24}$$

[7]The open string ends on the same D-brane.

[8]The same conclusion is reached if one follows the Dirac constrained quantization [7].

But the remaining commutation relations, involving the fermion operators S_n, remain *unmodified*. We will discuss the consequences of this for the construction of superspace in non-commutative geometry in section 4.

3.2 Unbroken supersymmetry

In this subsection, we will give a more explicit analysis of the unbroken supersymmetry living on the brane when $B \neq 0$. We will first identify the unbroken supercharges and then derive their supersymmetry algebra from the commutation relations (3.19)–(3.23).

To preserve light-cone gauge conditions (3.1) and (3.2) under the supersymmetry transformations (2.24), we have to perform a compensating κ-symmetry transformation. In order to present concrete formulae, we consider the case of type IIB and denote the supersymmetry parameters (with the convention that both have negative chirality, $\epsilon^A = -\Gamma^{11}\epsilon^A$) by

$$\epsilon^A = \begin{pmatrix} 0 \\ 0 \\ \eta^{(A)a} \\ \epsilon^{(A)\dot{a}} \end{pmatrix} . \tag{3.25}$$

The spacetime supersymmetry transformations (2.24) in light-cone gauge read

$$\begin{aligned} \delta S^{(1)a} &= \sqrt{2p^+}\eta^{(1)a} + \frac{1}{\alpha'\sqrt{p^+}}\partial X^I \gamma^I_{a\dot{a}}\epsilon^{(1)\dot{a}} , \\ \delta S^{(2)a} &= \sqrt{2p^+}\eta^{(2)a} + \frac{1}{\alpha'\sqrt{p^+}}\bar{\partial} X^I \gamma^I_{a\dot{a}}\epsilon^{(2)\dot{a}} , \\ \delta X^I &= \frac{i}{\sqrt{p^+}} S^{(A)a}\gamma^I_{a\dot{a}}\epsilon^{(A)\dot{a}} . \end{aligned} \tag{3.26}$$

The Noether supercharges for these symmetries are

$$\begin{aligned} Q^{(A)a} &= \sqrt{2p^+}\int_0^\pi \frac{d\sigma}{\pi} S^{(A)a} , \\ Q^{(1)\dot{a}} &= \frac{1}{\alpha'\sqrt{p^+}}\int_0^\pi \frac{d\sigma}{\pi}\partial X^I S^{(1)a}\gamma^I_{a\dot{a}} , \\ Q^{(2)\dot{a}} &= \frac{1}{\alpha'\sqrt{p^+}}\int_0^\pi \frac{d\sigma}{\pi}\bar{\partial} X^I S^{(2)a}\gamma^I_{a\dot{a}} \end{aligned} \tag{3.27}$$

and they satisfy $Q^{(2)} = NQ^{(1)}$. The sixteen unbroken supercharges are generated by the linear combinations

$$Q^a := \frac{1}{2}\left(Q^{(1)a} + N^{ba}Q^{(2)b}\right), \quad Q^{\dot{a}} := \frac{1}{2}\left(Q^{(1)\dot{a}} + N^{\dot{b}\dot{a}}Q^{(2)\dot{b}}\right). \tag{3.28}$$

Explicitly,

$$\begin{aligned} Q^a &= Q^{(1)a} = \sqrt{2p^+}\, S^a_0 . \\ Q^{\dot{a}} &= Q^{(1)\dot{a}} = \frac{1}{\sqrt{p^+\alpha'}}\sum_{n\neq 0}\left\{\gamma^i_{a\dot{a}}\left(S^a_{-n}\alpha^i_n + \sqrt{\alpha'}S^a_0 p^i\right) + \gamma^{i'}_{a\dot{a}}S^a_{-n}\alpha^{i'}_n\right\} . \end{aligned} \tag{3.29}$$

Now, we can use the commutation relations (3.19)–(3.23) to compute the algebra of the unbroken supercharges. It takes the standard form

$$\begin{aligned}\{Q^a, Q^b\} &= 2p^+\delta^{ab}\,,\\ \{Q^a, Q^{\dot a}\} &= \sqrt{2}\gamma^i_{a\dot a}p^i\,,\\ \{Q^{\dot a}, Q^{\dot b}\} &= 2H\delta^{\dot a\dot b}\,,\end{aligned} \tag{3.30}$$

with H being light-cone gauge hamiltonian (3.18). This shows that while the linear combination (3.28) of unbroken supersymmetry depends on the B-field, the form of the supersymmetry algebra is not deformed by the B-field. We will also discuss the consequences of this in section 4.

But before doing that, let us comment on how supersymmetry is realized in the RNS formalism. The quantization of the RNS fermions $\psi^i = (\psi^i_+, \psi^i_-)$ was carried out in [8, 7] and the following result was obtained [7]:

$$\{\psi^i_+(\sigma), \psi^j_+(\sigma')\} = \pi\eta^{ij}\delta(\sigma,\sigma')\,, \qquad \forall\sigma,\sigma' \tag{3.31}$$

$$\{\psi^i_-(\sigma), \psi^j_-(\sigma')\} = \pi\eta^{ij}\delta(\sigma,\sigma')\,, \tag{3.32}$$

$$\{\psi^i_+(\sigma), \psi^j_-(\sigma')\} = \pi((1+B)(1-B)^{-1})^{ij}\delta(\sigma,\sigma')\,, \quad \sigma,\sigma' \text{ on the boundary}\,, \tag{3.33}$$

with the boundary condition

$$\psi^j_+(1+B)_j{}^i = \psi^j_-(1-B)_j{}^i\,, \qquad i,j = 0,1,\ldots,p\,. \tag{3.34}$$

Note in particular that the $++$ and $--$ commutation relations are not modified. The boundary condition relates the left and right-moving modes for each NS and R sector,

$$\tilde{d}^i_n = d^k_n\left(\frac{1-B}{1+B}\right)_k{}^i, \qquad \tilde{b}^i_r = b^k_r\left(\frac{1-B}{1+B}\right)_k{}^i \tag{3.35}$$

and shows that (3.31) implies (3.32) and (3.33). The zero modes of (3.31) in the Ramond sector form the *usual* Clifford algebra acting on the ground state and represent the unbroken supersymmetry on the D-brane.

3.3 Moyal product

In this paper we use the oscillator quantization method to obtain our results. But the same conclusions are obtained if one uses conformal field theory methods. As was derived in [18], the world-sheet propagator for the bosonic fields X acquires a crucial new logarithmic term in the presence of a constant B-field. In the Sen-Seiberg limit, it is

$$\langle x^i(\tau)x^j(0)\rangle = i\frac{\Theta^{ij}}{2}\mathrm{sign}(\tau), \tag{3.36}$$

which by the radial ordering prescription, gives

$$[x^i(\tau), x^j(\tau)] = T\left(x^i(\tau^+)x^j(\tau) - x^i(\tau)x^j(\tau^+)\right) = i\Theta^{ij}. \tag{3.37}$$

The leading OPE for normal ordered (i.e. finite) bosonic open string vertex operators was shown [8, 2] to become the Moyal product.

On the other hand, since the fermion propagators satisfy first order differential equations, they do not carry any logarithmic branch cut. It is true that their boundary conditions modify their OPE; for instance, in type IIB:

$$\begin{aligned} \hat{\theta}^{(A)a}(\tau)\hat{\theta}^{(A)b}(\tau') &\sim \frac{\delta^{ab}}{\tau-\tau'}, \qquad A=1,2 \text{ (not summed)}, \\ \hat{\theta}^{(1)a}(\tau)\hat{\theta}^{(2)b}(\tau') &\sim \frac{N^{ab}}{\tau-\tau'}. \end{aligned} \tag{3.38}$$

However, there is no modification of their anti-commutators,

$$\{\hat{\theta}^{(A)a}, \hat{\theta}^{(B)b}\} = T\left(\hat{\theta}^{(A)a}(\tau^+)\hat{\theta}^{(B)b}(\tau) + \hat{\theta}^{(A)a}(\tau)\hat{\theta}^{(B)b}(\tau^+)\right) = 0\,, \tag{3.39}$$

which agrees with (3.22).

Let us remark on how the Moyal phases come out in the oscillators quantization method. Consider the zero momentum Fock-space ground state of the open bosonic string, $|0,0\rangle$. The tachyon state $|0,k\rangle$ is obtained by the insertion in the far past of the tachyon vertex operator, located at the world-sheet boundary (take for instance $\sigma = 0$)

$$V_0(k;\tau) =: e^{ik\cdot X(0,\tau)} : \,. \tag{3.40}$$

Explicitly, the state is

$$|0,k\rangle = \lim_{\tau\to+i\infty} e^{-i\tau} V_0(k;\tau)|0,0\rangle\,. \tag{3.41}$$

Since $V_0(k,\tau)$ is normal ordered, (3.41) is just

$$|0,k\rangle = e^{ik\cdot x_0}|0,0\rangle\,. \tag{3.42}$$

Observe that (3.42) satisfies the intuitive property that in the absence of a B field,

$$|0,k_1+k_2\rangle = e^{i(k_1+k_2)\cdot x_0}|0,0\rangle = e^{ik_1\cdot x_0}|0,k_2\rangle = e^{ik_2\cdot x_0}|0,k_1\rangle\,. \tag{3.43}$$

But when there is a constant background $B \neq 0$, the state (3.42) with momentum k will no longer satisfy the last three equalities in (3.43). Instead, the Moyal phase appears due to (3.20)

$$\lim_{\tau\to+i\infty} e^{-i\tau} V_0(k_2,\tau)|0,k_1\rangle = e^{ik_2\cdot x_0}e^{ik_1\cdot x_0}|0,0\rangle = e^{ik_1\wedge k_2}|0,k_1+k_2\rangle\,, \tag{3.44}$$

where $k_1 \wedge k_2 := \frac{1}{2}\Theta^{ij}k_{1,i}k_{2,j}$. This explains the appearance of the Moyal phases in interaction vertices. For example, for the three tachyon vertex

$$\langle 0,k_1|V_0(k_2,0)|0,k_3\rangle = \langle 0,k_1|e^{ik_2\wedge k_3}|0,k_2+k_3\rangle = e^{ik_2\wedge k_3}\delta(-k_1+k_2+k_3)\,. \tag{3.45}$$

4. Discussion: superspace and superfields

We have derived that for the non-commutative supersymmetric quantum field theories obtained from string theory, the fermionic variables θ continue to anti-commute. Motivated by this, we introduce the non-commutative superfield, $\Phi(x,\theta)$, to be a local function

$$\Phi(x,\theta) = \phi(x) + \theta\lambda(x) + \cdots \tag{4.1}$$

Obviously, it continues to have a finite Taylor expansion in θ. The Moyal product for superfields evaluated on the supermanifold completely ignores the fermionic coordinates, i.e.

$$\begin{aligned}\Phi^I(x,\theta) \star \Phi^J(x,\theta) &= e^{\frac{i\Theta^{ij}}{2}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}}\,\Phi^I(x,\theta)\Phi^J(y,\theta)\Big|_{x=y} \\ &= \phi^I(x)\star\phi^J(x) + \theta\left(\phi^I(x)\star\lambda^J(x) + \lambda^I(x)\star\phi^J(x)\right) + \cdots\end{aligned} \tag{4.2}$$

In order to be concrete, let us focus on the case of 4d $\mathcal{N}=1$ supersymmetry[9] It has supercharges $\{Q_\alpha, \overline{Q}_{\dot\alpha}\}$ and the supersymmetry algebra can be obtained by a convenient truncation of the 10d $\mathcal{N}=1$ supersymmetry algebra (3.30).[10] The realization of the supercharges as first order differential operators on the superspace $\{x,\theta,\bar\theta\}$ is the same as in the ordinary spacetime case. The same is true for the supersymmetric covariant derivatives, $\{D_\alpha, \overline{D}_{\dot\alpha}\}$. Supersymmetry transformations are generated by the usual shifts of the superspace variables:

$$\begin{aligned}&\theta \to \theta + \xi\,, \\ &x^\mu \to x^\mu + i\theta\sigma^\mu\bar\xi - i\xi\sigma^\mu\bar\theta\,,\end{aligned} \tag{4.3}$$

and the same sort of supersymmetric field transformations are obtained. The only modification comes from replacing the ordinary commutative product by the Moyal product for any product of x-dependent quantities.

Superfields can be introduced as field representations of the supersymmetry algebra. For example, chiral superfields satisfy $\overline{D}_{\dot\alpha}\Phi = 0$. Supersymmetric invariant actions are constructed in the same way, as integrals in superspace of some x-dependent superfield expression. The most general lagrangean that can be built from scalar chiral superfields takes the form

$$\mathcal{L} = \int d\theta^2 d\bar\theta^2\,\mathcal{K}(\overline{\Phi},\Phi) + \int d\theta^2\,\mathcal{W}(\Phi) + \text{h.c.} \tag{4.4}$$

[9] We follow the conventions of [19].

[10] Properly, the algebra (3.30) was derived in light-cone gauge. Here we extend its 4d $\mathcal{N}=1$ subalgebra to its covariant off-shell formulation.

We will still refer to $\mathcal{K}(\overline{\Phi}, \Phi)$ as the Kähler potential, simply because we still have the symmetry

$$\mathcal{K}(\overline{\Phi}, \Phi) \to \mathcal{K}(\overline{\Phi}, \Phi) + F(\Phi) + \overline{F}(\overline{\Phi})\,. \tag{4.5}$$

It is easy to show that the auxillary fields are nonpropagating and can be eliminated to give a supersymmetric action in terms of on-shell states only. This apparently differs from the situation in some higher derviative actions, e.g. in $N = 2$ SYM with nonholomorphic corrections, where the auxillary fields can become propagating.[11]

Observe that, for chiral superfields, one still has the powerful notion of a holomorphic superpotential $\mathcal{W}(\Phi)$, which only requires a chiral θ-integral to be included in the lagrangean. Also, one can formulate Feynman super-graphs for (4.4). All the quadratic divergences are cancelled out manifestly. It is easy to show that any perturbative quantum corrections to the effective action will be an integral over the whole superspace. Then, modulo the usual subtleties arising from massless fields, the perturbative non-renormalization of the F-term still holds for non-commutative supersymmetric field theories.[12] It has been argued in [22] that noncommutative Wess-Zumino model is renormalizable.

Obviously, the most interesting case is non-commutative supersymmetric Yang-Mills, where the Ward identities of the non-commutative gauge symmetry and its sensitive embedding in a consistent superstring theory suggests that it is a well defined quantum theory. But we leave this issue for future studies.

In this letter, we have shown that the manifestly supersymmetric formulation of the non-commutative supersymmetric field theories, obtained from string theory in flat spacetime and constant B-field backgrounds, shares a large number of the features in common with the ordinary supersymmetric field theories. The notion and construction of superspace and superfield is the same, without additional deformation of the fermionic variables θ. In more general string backgrounds, one may get a more non-trivial "Moyal product" for the fermionic coordinates of the relevant superspace. It would be interesting to know which are the admissible "Moyal products" for a given supermanifold and whether and how they arise in string theory.

Manifest supersymmetric formulations have proven to provide useful and powerful concepts, such as holomorphy and non-renormalization theorems, for analyzing different quantum aspects of the supersymmetric field theories. It is quite reasonable that a manifestly supersymmetric treatment of non-commutative supersymmetric field theories will also shed light on the dynamics of these theories.

[11]We thank J.P. Derendinger for useful discussions and comments on this.

[12]See [20, 21] for a recent discussion of IR effects in non-commutative field theories. A $1/p \circ p$ singularity in the two point function (arising from the 1 loop non-planar diagram) was found in the non-commutative theory. In particular, it was shown that a light mode is generated whose mass depends on the coupling of the theory [20]. They also show that no new pole is generated in the supersymmetric case. This can also be verified easily by doing a simple Feynman super-graph calculation.

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